Odd-Intersections Lemma: Let K be a virtual knot diagram drawn classically on a surface S of minimal genus and let α be a simple closed non-disconnecting curve on S such that $|\alpha \cap K|$ is even. Then K has an odd intersection with respect to S.

Def: For a virtual knot diagram K, let $g(K)$ be the genus of the minimal possible genus surface it can be classically drawn on.

Suppose

$$K_0 \sim K_1 \sim \dots \sim K_n$$

is a sequence of virtual knot diagrams connected by Reidemeister moves such that

K_0, K_n are classical

some K_i is not classical.

There must be a subsequence

$$A \sim K_1' \sim K_2' \sim \dots \sim K_m' \sim B$$

such that $g(K_1') = g(K_2') = \dots = g(K_m') = g(A)+1 = g(B)+1$

Then $A, K_1', \dots, K_m', B$ can all be drawn classicaly on the same surface S.

Moreover there are simple, closed, non-disconnecting curves α, β on S such that α does not intersect A, β does not intersect B.

~~Specifically~~

$|\alpha \cap K_i'|$ even

$|\beta \cap K_i'|$ even

for all i

$|\alpha \cap B|$ even

$|\beta \cap A|$ even

~~For~~ α defines a parity a for knots drawn on S
β defines a parity b for knots drawn on S

$a \vee b$ is a weak parity for knots drawn on S.

$a \vee b = 1$ if $a = 1$ or $b = 1$.

For a parity p let pr_p be the projection with respect to p.

$$
\begin{array}{ccccccccc}
A = pr_a(A) & \sim & pr_a(K_1') & \sim & \cdots & \sim & pr_a(K_m') & \sim & pr_a(B) \\
 & & & & & & & & \parallel \\
pr_{a\vee b}(A) & \sim & pr_{a\vee b}(K_1') & \sim & \cdots & \sim & pr_{a\vee b}(K_m') & \sim & pr_{a\vee b}(B) \\
\parallel & & & & & & & & \\
pr_b(A) & \sim & pr_b(K_1') & \sim & \cdots & \sim & pr_b(K_m') & \sim & pr_b(B) = B
\end{array}
$$

since $c(A) < c(K_1')$, $c(B) < c(K_m')$